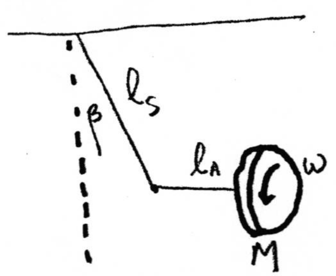
ID Number ________ **PART I** Score ________

PHYSICS DEPARTMENTAL EXAM – SPRING 2004

SECTION 1:

PROBLEM 1.

1. A solid disc of mass M and radius R is rotating at an angular frequency ω. As shown in the figure, the disc is attached to an axle of length ℓ_A which is being held up by a string of length ℓ_s. If the disc executes uniform precession in the horizontal plane and the axle is horizontal, find an equation for the angle β that the string makes with the vertical axis



Solution Prob. 1

Dept. Exam Solutions SP04

Solution 1

Moment of Inertia

$I = \frac{1}{2} M R^2$ for the disc. We can write down three equations, from force considerations (2) and from torque (1). In detail

$T \cos\beta = Mg$ (Force in vertical direction)

$T \sin\beta = M(L \sin\beta + l)\Omega^2$ (horizontal)

if Ω is the precession rate

and finally, torque equals rate of change of angular momentum. Here

$$\frac{dL}{dt} = \Omega L = I\omega\Omega$$

This must equal the torque (with respect to ~~point~~ O). the center of the circle along which the disc is precessing

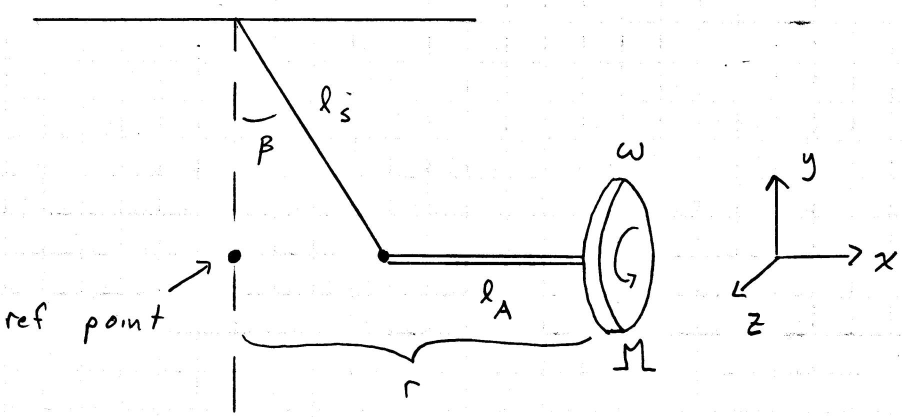
$$I\omega\Omega = Mg(l + L\sin\beta) - T L \cos\beta \sin\beta$$
$$= Mgl$$

Solving these equations gives $\Omega = \frac{Mgl}{I\omega}$

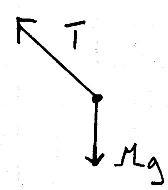
and

$$M(L\sin\beta + l)\Omega^2 = Mg\tan\beta$$

A solid disc of mass M and radius R is rotating at an angular frequency ω. A shown in the figure, the disc is attached to an axle of length l_A which is being held up by a string l_s. If the disc executes uniform precession in the horizontal plane and the axle is horizontal, find an equation for the angle β that the string makes with the vertical axis.



First, balance the forces:



two forces

$$\vec{T} + Mg(-\hat{y}) = \frac{Mv^2}{r}(-\hat{x}) = M\Omega^2(l_s \sin\beta + l_A)(-\hat{x})$$

$$\Rightarrow \quad T\cos\beta = Mg \qquad (y\text{-comp})$$

$$T\sin\beta = M\Omega^2 \qquad (x\text{-comp})$$

so,

$$\tan\beta = \frac{\Omega^2}{g}(l_s \sin\beta + l_A)$$

However, we are not done. We can now balance the torques to elliminate Ω.

Assumption: we can neglect the mass of the axle compared to the mass of the disc, i.e. $m_A \ll M$.

Balance torques:

$$\vec{N} = \sum_i \vec{r}_i \times \vec{F}_i$$

$$= \vec{r}_1 \times \vec{T} + \vec{r}_2 \times \vec{F}_g$$

$$\vec{r}_1 = l_s \sin\beta \; \hat{x}$$

$$\vec{r}_2 = (l_s \sin\beta + l_A)\, \hat{x}$$

$$\vec{T} = T_x(-\hat{x}) + T_y(\hat{y})$$

$$\vec{F}_g = Mg(-\hat{y})$$

$$\vec{N} = l_s \sin\beta \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & 0 \\ -T_x & T_y & 0 \end{vmatrix} - Mg\,(l_s \sin\beta + l_A)\underbrace{(\hat{x} \times \hat{y})}_{\hat{z}}$$

$$= \hat{z}\left\{ Mg\, l_s \sin\beta - Mg\,(l_s \sin\beta + l_A) \right\}$$

$$= -Mg\, l_A\, \hat{z}$$

Thus, the precession direction is counter clockwise (as viewed from above). We know $\vec{\Omega} = \Omega(+\hat{y})$.

Now use $\vec{N} = \dfrac{d\vec{L}_{tot}}{dt}$

$$\vec{L}_{tot} = \vec{L}_{cm} + \vec{L}'$$

$\vec{L}' = I\omega\, \hat{x}$, ang mom. due to spin about axle

$\vec{L}_{cm} = Mr^2\Omega\, \hat{y}$, ang mom. due to precession of center of mass of the disk.

Note: $\hat{x}, \hat{y}$ and $\hat{z}$ are body fixed coordinates but only

$$\frac{d\hat{y}}{dt} = 0 \qquad \text{(always pointed up).}$$

That is $\hat{x}$ and $\hat{z}$ change directions as a function of time.

$$\Rightarrow \quad \frac{d\vec{L}_{tot}}{dt} = \underbrace{\cancel{\frac{d\vec{L}_{cm}}{dt}}}_{0} + \frac{d\vec{L}'}{dt}$$

since $\vec{L}_{cm}$ is constant

$$= \underbrace{\cancel{\left(\frac{d\vec{L}'}{dt}\right)_{Body}}}_{0} + \vec{\Omega} \times \vec{L}'$$

since $\vec{L}'$ has a constant magnitude

$$= I\omega\Omega\,(\hat{y} \times \hat{x})$$

$$= I\omega\Omega\,(-\hat{z})$$

Thus, $\quad Mg\,l_A = I\omega\Omega$

$$\Omega = \frac{Mg\,l_A}{I\omega} = \frac{2g\,l_A}{\omega R^2}$$

Our equation for β becomes:

$$\boxed{\tan\beta = (l_s \sin\beta + l_A)\,\frac{4g\,l_A^{\,2}}{\omega^2 R^4}}$$

This is trancendental so this cannot be solved analytically.