QUAL equation sheet

1. MECHANICS

1.1. Calculus of Variations

minimize line integral:

$$D = \int \sqrt{(dx)^2 + (dy)^2 + (\frac{dz}{dx})^2(dx)^2} = \int dx(L(x, y, \dot{y}))$$

1.2. Finding normal modes

$$|-\omega^2 T + V| = 0$$

1.3. Constraint Forces

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \sum_a \lambda_a \frac{\partial F_a}{\partial q}$$

1.4. Small Oscillations

Plug in $\theta = \theta_0 + \delta\theta$ and expand for small $\delta\theta$. Oscillations are only stable if $\ddot{\delta\theta} = -\omega\delta\theta$
Only for circular orbits:

$$k_{eff} = \frac{\partial^2 V}{\partial x^2}$$

1.5. Rotating bodies

$$\tau_{\vec{ext}} = \frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L}$$

$$I_1 \cdot \omega_1 - (I_2 - I_3)\omega_2\omega_3 = N_1$$

$$I_2 \cdot \omega_2 - (I_3 - I_1)\omega_1\omega_3 = N_2$$

$$I_3 \cdot \omega_3 - (I_1 - I_2)\omega_1\omega_2 = N_3$$

1.6. Non-inertial Ref frames

$$\vec{F} = mg\hat{z} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m(\vec{\omega} \times \vec{v})$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\omega = \Omega(-\sin\theta\hat{y} + cos\theta\hat{z})$$

$$(\frac{d\vec{Q}}{dt})_{fixed} = (\frac{d\vec{Q}}{dt})_{rotating} + \vec{\omega} \times \vec{Q}$$

1.7. Torque

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} = \vec{L} \cdot \vec{\omega}$$

$$L = I\Omega$$

1.8. Moments of inertia

Rod about the center:	$(1/12)MR^2$
Hoop about the center:	MR^2
Disk about the center:	$(1/2)MR^2$
Sphere about the center:	$(2/5)MR^2$

$$I_{\|} = I_{CM} + Md^2$$

1.9. String wave

$$c = \sqrt{\frac{\tau}{\mu}}$$

$$F = \tau[\frac{du}{dx}_+ - \frac{du}{dx}_-]$$

1.10. Central Force Motion

$$U_{eff} = \frac{l^2}{2\mu r^2} + U(r)$$

$$l = \mu r^2 \dot{\phi}$$

-Circular orbit requires $\frac{\partial U_{eff}}{\partial r}|_{r_0} = 0$

-Can Taylor expand $U_{eff}(r) = U_0 + \frac{1}{2}(r - r_0)^2 U''_{eff}(r_0)$

Now $k_{eff} = \frac{\partial^2 V}{\partial x^2}$

Can compare orbital frequency, $\dot{\phi}$ with ω to get rate of precession of the perihelion. Will change by an angle $\theta = T\omega$ where $T = 2\pi/\omega$.

-Substitute $r = \frac{1}{s}$

$$\frac{l}{m}\frac{d}{d\phi} = \frac{1}{s^2}\frac{d}{dt}$$

-Conserve energy, then $\frac{dE}{dt} = 0$

Energy	ϵ	Type
0	1	parabola
> 0	> 1	hyperbola
$\frac{mk^2}{2l^2} < E < 0$	< 1	ellipse
$\frac{-mk^2}{2l^2}$	0	cicle

-Semimajor axis $a = \frac{-GMm}{2E}$
-Closest approach $a(1-\epsilon)$ farthest approach $a(1+\epsilon)$
-Foci at $\pm\epsilon a$

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta}$$

$$\alpha = a(1-\epsilon^2) = \frac{l^2}{mk}$$

$$\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$$

-Virial theorem: $<T> = -\frac{1}{2} <V>$

$$<E> = <T> + <V> = \frac{1}{2} <V> = \frac{-k}{2a}$$

Impact parameter, b, velocity v_0. If zero of energy is at ∞, set $T = -V$.

$$R = \sqrt{(vt)^2 + b^2}$$

$$L_0 = mv_0 R sin\theta = mv_0 b$$

1.11. Poisson Bracket

$$\{A, B\} = \sum_j \left(\frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j}\frac{\partial B}{\partial q_j}\right)$$

$$\{A, H\} = \dot{A}$$

If $\{H, F_1\} = 0$, then F_1 is a constant of the motion
if $\{F_1, F_2\} = 1$, There are no further constants

For a canonical transformation, $\{Q, P\} = 1$.
To find the generating function F, set $P = -\frac{\partial F}{\partial Q}$, plug into Hamiltonian, replacing $p = \frac{\partial F}{\partial q}$ and $x = x(P, Q)$ which gives you $K(P, Q)$.
$\dot{P} = -\frac{\partial K}{\partial Q}$ and $\dot{Q} = \frac{\partial K}{\partial P}$ (look up S02-13)

1.12. Magnetic Field (non-relativistic)

$$L = \frac{e}{c}\vec{v} \cdot \vec{A} - e\Phi$$

$$H = \dot{x}p_x + \dot{y}p_y - L = e\Phi$$

1.13. Hamilton's Equations

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}}$$

$$H = p_\phi \dot{\phi} - L$$

$$\dot{q_i} = \frac{\partial H(q_i, p_i)}{\partial p_i}$$

$$\dot{p_i} = -\frac{\partial H(q_i, p_i)}{\partial q_i}$$

1.14. Misc

$$x = x_o + v_0 t + \frac{1}{2}at^2$$

$$v_f^2 = v_0^2 + 2ax$$

$$v_f = v_0 + at$$

Adiabatic Invariant (for mechanical mirror problem(S06-12))

$$2\pi I = \int_{-y(x)}^{y(x)} mv_y dy + \int_{y(x)}^{-y(x)} -mv_y dy = 4my(x)v_y$$

Next plug in result to get $E(I, v_x)$ and solve to get real v_x. Initial conditions define I.

2. E&M

2.1. Relativity, $(+,-,-,-)$

$$v_x = \frac{v_x' + v}{1 + \frac{v_x v}{c^2}}$$

$$v_y = \frac{\gamma v_y}{1 + \frac{v_x v}{c^2}}$$

$$a_\parallel = \frac{a_\parallel'}{\gamma}$$

$$u^i = (\gamma c, \gamma v)$$

$$p^i = (\gamma mc, \gamma mv) = (\frac{E}{c}, \vec{p})$$

$$A^i = (\phi, \vec{A})$$

$$j^i = (c\rho, \vec{J}$$

$$x^{\mu'} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} x^\mu$$

$$\vec{E_\perp}' = \gamma[\vec{E_\perp} + \beta \times \vec{B_\perp}]$$

$$\vec{B_\perp}' = \gamma[\vec{B_\perp} - \beta \times \vec{E_\perp}]$$

Scalars (same in every frame) include $B^2 - E^2$ and $\vec{E} \cdot \vec{B}$.

$$L = \frac{-mc^2}{\gamma} - e\phi + \frac{e}{c}\vec{v} \cdot \vec{A}$$

$$\vec{p} = -\frac{\partial L}{\partial v} = \gamma mv + \frac{e}{c}\vec{A}$$

$$E = pv - L = \gamma mc^2 + e\phi = \sqrt{(pc)^2 + (mc^2)^2}$$

For reactions/collisions must conserve p^μ_{Total}. In any frame for each particle, $p^\mu \cdot p_\mu = -m^2c^2 = -\frac{E^2}{c^2} + \vec{p}^2$

2.2. Undergrad (Baby stuff)

$$\vec{J} = \sigma \vec{E}$$

$$V = \int \vec{E} \cdot \vec{dl}$$

$$V = IR$$

$$P = I^2 R$$

$$\vec{F} = q[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}]$$

$$\vec{F} = \frac{I}{c}\vec{l} \times \vec{B}$$

$$emf = \frac{1}{c}\frac{d\Phi}{dt} = L\dot{I}$$

2.3. E and B

Monochromatic plane wave:

$$\vec{E} = \vec{E}_o e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r})$$

$$\oint \vec{A} \cdot \vec{dl} = \int \vec{B} \cdot d\vec{A}$$

$$\Omega_L = \frac{eB}{\gamma mc}$$

$$\Delta(\frac{mv_\perp^2}{2B}) = 0$$

$$v_{Drift} = \frac{\vec{E} \times \vec{B}}{|\vec{B}|^2}$$

$$< g(t)h(t) >= \frac{1}{2} Re[gh^*]$$

$$\frac{c}{\omega}\vec{k} \times \vec{E} = \vec{B}$$

2.4. Energy and Momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$$W = \frac{1}{8\pi}(E^2 + B^2)$$

$$\frac{\vec{S}}{c} = \frac{1}{4\pi c}(\vec{E} \times \vec{B})$$

$$\vec{E} \cdot \vec{J} + \frac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0$$

$$\rho \vec{E} + \frac{1}{c}\vec{J} \times \vec{B} + \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} + \vec{\nabla} \cdot \vec{T} = 0$$

$$W_e = \frac{1}{2}\int dV[\rho\phi]$$

$$W_b = \frac{1}{2c}\int dV(\vec{J} \cdot \vec{A})$$

For outward pointing $d\vec{A}$:

$$\tilde{\tilde{T}} = \frac{1}{4\pi}[\tilde{E}\tilde{E} + \tilde{B}\tilde{B} - \frac{1}{2}\tilde{\tilde{I}}(E^2 + B^2)]$$

$$d\vec{F} = \tilde{\tilde{T}} \cdot d\vec{A}$$

Alternative notation (careful):

$$P = \frac{dU}{dV} = \frac{|E|^2}{8\pi}$$

$$F = \int P dA$$

2.5. Dipoles

Potential from a charge:

$$\phi(r) = \frac{q}{|r - r_1|} \approx \frac{q}{r}(1 + \frac{r_1}{r}\cos\theta)$$

$$\phi(r) = \int dV' \frac{\rho(r)}{|r - r'|} \approx \frac{Q}{r} + \frac{\vec{r} \cdot \vec{p}}{r^3}$$

$$Q = \int dV' \rho(r')$$

$$\vec{p} = \int dV' r' \rho(r') \approx qr$$

$$\vec{E} = \frac{3\vec{r}(\vec{r}\cdot\vec{p}) - r^2\vec{p}}{r^5}$$

$$\vec{\tau} = \vec{p} \times \vec{E_{ex}}$$

For the hemisphere problem(two halves at diff pot, time dep.): find σ using ϕ^{in} and ϕ^{out}

$$\vec{p} = \int d\Omega R \cos\theta \sigma(theta)$$

Dipole and magnetic monopole of e^- colliding with e^- disappears, dipole of e^- and e^+ survives(S06-13)
Dipole of a dielectric sphere in an EM field:

$$\vec{p} = V\vec{P} = \frac{V}{4\pi}(D - \epsilon_0 E) = \frac{V}{4\pi}(\epsilon - \epsilon_0)\vec{E_{inside}}$$

In the far zone region(F01-16):

$$\vec{A} = \frac{q}{rc}\dot{\vec{p}}(t - \frac{r}{c})$$

$$\vec{B} = \frac{q}{rc^2}\ddot{\vec{p}} \times \hat{r}$$

$$\vec{E} = \frac{q}{rc}(\ddot{\vec{p}} \times \hat{r}) \times \hat{r}$$

$$\vec{S} = \frac{c}{4\pi}(\frac{q}{rc^2})^2|\ddot{\vec{p}} \times \hat{r}|$$

Now replace $\ddot{\vec{p}} = e^{ikr}\vec{p}$

$$\frac{dP}{d\Omega} = \frac{r^2}{2}\hat{r}Re(\vec{E} \times \vec{H^*}) = \frac{q^2}{4\pi c^3}|\ddot{\vec{p}} \times \hat{r}|^2 = \frac{q^2}{4\pi c^3}|\ddot{\vec{p}}|^2 \sin^2\theta$$

$$\vec{A} = \frac{\vec{m} \times \vec{r}}{r^3}$$

$$\vec{m} = \frac{1}{2c}\int dV' \vec{r'} \times \vec{J}(r') \approx \frac{IA}{c}$$

$$\vec{B} = \frac{3\vec{r}(\vec{r}\cdot\vec{m}) - r^2\vec{m}}{r^3} = \frac{m}{r^3}(2cos\theta\hat{r} + sin\theta\hat{\theta}))$$

$$\vec{F} = \vec{\nabla}[\vec{m} \cdot \vec{B_{ex}}]$$

$$\vec{\tau} = \vec{m} \times \vec{B_{ex}}$$

2.6. Radiation from a dipole

$$\frac{d < P >}{d\Omega} = \frac{ck^4}{8\pi}|p|^2 sin^2\theta$$

$$< P >= \frac{\omega^4|p|^2}{3c^3} = \frac{2}{3}\frac{|\ddot{p}|^2}{c}$$

$$< S >= \frac{1}{2}Re[\vec{E} \times \vec{B^*}] = \hat{r}\frac{ck^4}{8\pi}\frac{|\hat{r} \times \vec{p}|^2}{r^2}$$

Thompson scattering (EM wave on e^-) gives $m\ddot{\Delta r} = e\vec{E}$, solve for Δr to get a dipole.

2.7. Larmor formula(F02-15) and (S98-14)

$$P_{rad} = \frac{2e^2a^2}{3c^3} = -\frac{dE}{dt}$$

$$a = \frac{F}{m} = v\omega_L = v\frac{eB}{mc}$$

Relativistically:

$$P_{rad} = -\frac{2e^2}{3m^2c^3}\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}$$

$$t = \gamma\tau$$

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v} \times \vec{B}$$

2.8. Free and Bound currents/charges

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

$$\vec{J_b} = \frac{\partial\vec{P}}{\partial t} + c(\vec{\nabla} \times \vec{M})$$

$$\vec{K_b} = -c(\hat{n} \times \vec{M})$$

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f$$

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J_f} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$

$$\vec{H} = \vec{B} - 4\pi\vec{M}$$

$$\vec{P} = \chi_e\vec{E}$$

$$\vec{M} = \chi_m\vec{H}$$

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$

Dielectric gap in a wire, get J_f

2.9. Boundary Conditions

$$(D_2 - D_1) \cdot \hat{n} = 4\pi\sigma_f$$

$$\hat{n} \times (E_2 - E_1) = 0$$

$$(B_2 - B_1) \cdot \hat{n} = 0$$

$$\hat{n} \times (H_2 - H_1) = \frac{4\pi}{c}\vec{K}_f$$

-No surface charge for vacuum-dielectric boundary
-Do get surface charge for vacuum-conductor
-$E = 0$ in a conductor
-Neutral sphere, $\int \sigma dA = \int \rho dV$
-Only use B_0 term (outside potential expansion) when there is internal charge.
-Remember to include $\phi(r) = \frac{Q}{r} + \frac{\vec{r}\cdot\vec{p}}{r^3}$

-No surface current for discontinuous μ
-Do get surface current for vacuum-superconductor
-$B = 0$ in a superconductor

Moving sphere, zero Coulomb force , $\vec{F} = 0$

2.10. Maxwell's Equations

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\nabla^2\phi = 4\pi\rho$$

$\nabla \cdot E$	$4\pi\rho$	$\frac{\rho}{\epsilon}$
$\nabla \cdot B$	0	0
$\nabla \times E$	$-\frac{1}{c}\frac{\partial B}{\partial t}$	$\frac{\partial B}{\partial t}$
$\nabla \times B$	$\frac{4\pi}{c}J + \frac{1}{c}\frac{\partial E}{\partial t}$	$\mu_0 J + \mu_0\epsilon_0\frac{\partial E}{\partial t}$

2.11. Dispersion

$$k^2 = \frac{\omega^2}{c^2}\epsilon(\omega)\mu(\omega)$$

$$n(\omega) = \frac{c}{|v_{ph}|} = \sqrt{\epsilon(\omega)\mu(\omega)}$$

$$v_{ph} = \frac{\omega}{k}$$

$$v_g = \frac{d\omega}{dk}$$

-For an EM wave incident on a plasma (F03-13), continuous E_t, $B = \frac{c}{w}\vec{k} \times \vec{E}$, but in the plasma, $ck \neq \omega$, set up scattering to get that $R = 1$ because $n(\omega) = \sqrt{\epsilon}$ is imaginary, all of the energy is reflected at the boundary, wave is exponentially damped.

2.12. Reflection/Refraction

Boundary conds. for EM wave incident on a conductor give $E_R = -E_I$ and $B_R = B_I$

$$R_\perp = |\frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)}|^2$$

$$R_\parallel = |\frac{\tan(\theta' - \theta)}{\tan(\theta' + \theta)}|^2$$

Brewster's angle

$$\theta + \theta' = \frac{\pi}{2}$$

Total internal reflection:

$$\frac{n_1}{n_2}\sin\theta > 1$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

2.13. Waveguides (F04-13)

For TM modes: $E_z = 0$ on the edges

$$(\nabla_t^2 + [(\frac{\omega}{c})^2 - k^2])E_z(x, y) = 0$$

$$E_z = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

For TE modes:

$$(\nabla_t^2 + [(\frac{\omega}{c})^2 - k^2])B_z(x,y) = 0$$

$$B_z = \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b}$$

$$\omega_{mn}(k) = \sqrt{c^2k^2 + (\frac{m\pi c}{a})^2 + (\frac{n\pi c}{b})^2}$$

Cutoff Frequency:

$$0 < k^2c^2 = -\gamma^2c^2 + \omega^2$$

2.14. Method of Images

For a charge q that is a distance D away from a grounded sphere of radius a:

$$q' = -q(\frac{a}{D})$$

$$\vec{d} = \vec{D}(\frac{a}{D})^2$$

$$\Phi(r) = \frac{q}{4\pi r}$$

If not grounded, there is also an image charge at the origin, $q'' = Q - q'$, and the sphere has total charge Q

To find work to move charge from ∞,

$$E(r) = -\frac{\partial\phi_{image}}{\partial r}|_{r=D}$$

$$W = -q\int_D^\infty drE(r)$$

2.15. Constructing a potential given B.C's, Rectangular

$$V(x,y) = \sum_{n,m} A_{nm}sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})sinh(\gamma_{n,m}z)$$

$$\gamma_{nm}^2 = \alpha^2 + \beta^2$$

$$(sinh(\gamma_{nm}c))A_{nm} = \frac{4}{ab}\int_0^a dx\int_0^b dyV(x,y)sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})$$

$$\phi(x,y,z) = V(x,y)\frac{sinh(\gamma z)}{sinh(\gamma c)}$$

2.16. Cylinders (S02-16)

$$B_z(r,t) = \sum_n A_nJ_0(\frac{r}{R}\chi_{0n})e^{-\nu_n t}$$

$$\nu_n = \frac{c^2\chi_{0,n}^2}{4\pi rR^2}$$

$$\Phi(r,\theta,z) = \sum_{ln} J_l[\frac{x_{ln}}{a}r]sinh(\frac{x_{ln}}{a}z)[A_{ln}\cos l\theta + B_{ln}\sin l\theta]$$

$$A_{ln} = \frac{1}{\pi(\frac{a^2}{2})J_{l+1}^2(x_{ln})}\int_0^{2\pi} d\theta cos(l\theta)\int_0^a rdrJ_l(\frac{x_{ln}r}{a})V(r,\theta)$$

2.17. Spheres

$$\Phi = \sum_{lm}[A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}]Y_{lm}(\theta,\phi)$$

$$a^lA_{lm} = \int\int d\Omega Y_{lm}V(\theta,\phi)$$

2.18. Quasi-static B field near a conductor(F01-15) surface $z = 0$

$$\nabla^2\vec{B} = \frac{4\pi\sigma\mu}{c^2}\frac{\partial\vec{B}}{\partial t}$$

$$\delta = \sqrt{\frac{c^2}{\omega\sigma\mu}}$$

$$\vec{H_{vac}} = H_0e^{-i\omega t}\hat{x}$$

$$\vec{H_{cond}} = H(z)e^{-i\omega t}\hat{x}$$

$$k^2 = \frac{-i\omega 4\pi\sigma\mu}{c^2}$$

$$(\frac{d^2}{dz^2} + i\mu\sigma\omega)H(z) = 0$$

$$H(z) = H_0e^{(i-1)z/\delta}$$

$$E = \frac{1}{\sigma}\nabla\times H = \frac{\hat{y}}{\sigma}\frac{dH_x}{dz}$$

$$< \frac{PowerDiss}{area} >= \int_0^\infty \sigma < E^2 > dx = \frac{1}{2\sigma\delta}|\frac{c}{4\pi}H_0|^2$$

2.19. MISC

Gauge from a current carrying wire:

$$\vec{A} = -\frac{2}{c} ln(\frac{r}{r_0})\hat{z}$$

Gauge from $\vec{B} = B_0\hat{z}$ in spherical coords:

$$A_\phi = \frac{1}{2}Br\sin\theta\hat{\phi}$$

in cartesian:

$$A = Bx\hat{y}$$

$$A = \frac{1}{2}B(-y\hat{x} + x\hat{y})$$

$$\hat{\theta} = \frac{x\hat{y} - y\hat{x}}{r}$$

in cylindrical:

$$A = \frac{B}{2}r\hat{\phi}$$

Circular Polarization $|b_1| = |b_2|$, plus is LH, minus is RH:

$$\vec{E_0} = b_1 e^{-i\alpha}(\hat{x} \pm i\hat{y})$$

Cross section for light scattering off dipoles?(S00-19):

$$\sigma = \frac{P_{rad}}{\frac{c}{4\pi}|E_0|^2}$$

3. QUANTUM

3.1. Misc

$$J(x,t) = \frac{i\hbar}{2m}(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x})$$

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\Delta_x\Delta_p = \Delta E\Delta t \geq \frac{\hbar}{2}$$

$$\sigma_a^2\sigma_b^2 \geq (\frac{1}{2i} < [\hat{A},\hat{B}] >)^2$$

$$\frac{d}{dt} < Q >= \frac{i}{\hbar} < [H,Q] > + < \frac{\partial Q}{\partial t}$$

$$[x,p] = i\hbar$$

$$\Delta(\frac{d\psi}{dx}) = \frac{2mV_0}{\hbar^2}\psi(0)$$

$$\psi(r_1,r_2) = \pm\psi(r_2,r_1)$$

$$\psi_\pm = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)]$$

3.2. Hydrogen Atom

$$V_{eff} = V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$$

$$E_n = -\frac{1}{2}mc^2\alpha^2\frac{1}{n^2} = -13.6eV\frac{1}{n^2}$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}cos\theta$$

$$Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3cos^2\theta - 1)$$

$$R_{10} = \frac{2}{a^{3/2}}e^{-r/a}$$

$$R_{20} = \frac{1}{\sqrt{2}}\frac{1}{a^{3/2}}(1 - \frac{1}{2}\frac{r}{a})e^{-r/2a}$$

$$R_{21} = \frac{1}{\sqrt{24}}\frac{1}{a^{3/2}}\frac{r}{a}e^{-r/2a}$$

3.3. Harmonic Oscillator

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega})$$

$$|n >= \frac{1}{\sqrt{n!}}(a^\dagger)^n|o >$$

$$a^\dagger|n >= \sqrt{n+1}|n+1 >$$

$$a|n >= \sqrt{n}|n-1 >$$

$$H = \hbar\omega(a^\dagger a + \frac{1}{2})$$

$$[a, a^\dagger] = 1$$

$$E = \hbar\omega(n_x + n_y + n_z + \frac{3}{2})$$

$$\psi_0 = (\frac{m\omega}{\pi\hbar})^{3/4}e^{-\frac{m\omega x^2}{2\hbar}}$$

(S02-18) For 2 oscillators with $H_{int} = ax_1x_2$ change variables to $x_1 = \frac{1}{\sqrt{2}}(y_1 + y_2)$ and $x_2 = \frac{1}{\sqrt{2}}(y_1 - y_2)$

Ground State (3d)

$$\psi = (\frac{m\omega}{\pi\hbar})^{3/4}exp[\frac{-m\omega}{2\pi}(x^2 + y^2 + z^2)]$$

3.4. Spin

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

All Eigenvalues are ± 1

$$|z, + >= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |z, - >= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|x, + >= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, |x, - >= \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$|y, + >= \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}, |y, - >= \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}, \sigma_y = \frac{1}{\sqrt{2}}\begin{pmatrix} & -i & \\ i & & -i \\ & i & \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

3.5. Angular Momentum

$$[L_x, L_y] = i\hbar L_z$$

$$L_\pm = L_x \pm iL_y$$

$$L_\pm|l, m >= \hbar\sqrt{s(s+1) - m(m\pm 1)}|l, m >$$

3.6. Time Independent Perturbation Theory

$$\Delta E_1 =< 0|H'|0 >$$

$$\Delta E_2 = \sum_n \frac{|< n|H'|0 >|^2}{E_n - E_0}$$

$$\psi_0^1 = \sum_n \frac{< n|H'|0 >}{E_n - E_0}\psi_n$$

3.7. Time Dependent Perturbation Theory

$$C_n^{(1)} = -\frac{i}{\hbar}\int_{t_0}^{t} dt' exp[-i(\frac{E_n - E_i}{\hbar})t'] < n|V(t')|i >$$

3.8. Variational Method

-Use a trial wavefunction with parameter a
-Normalize w.f.
-Find $\bar{H} =< \psi|H|\psi >$
-Set $\frac{\partial \bar{H}}{\partial a} = 0$

3.9. Rotation

$$d^j_{m,m'}(\beta) =< j, m'|e^{\frac{-i\hat{n}\cdot\vec{J}\beta}{\hbar}}|j, m' >$$

$$D^j_{m,m'}(\alpha,\beta,\gamma) = e^{-(m'\alpha+m\gamma)} d^j_{m,m'}(\beta)$$

$$e^{-i\frac{\vec{\sigma}}{2}\cdot\hat{n}\theta} = \cos\frac{\theta}{2}\tilde{I} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\theta}{2}$$

$$d^{1/2}(\beta) = \begin{pmatrix} \cos(\beta/2) - in_z\sin(\beta/2) & (-in_x - n_y)\sin(\beta/2) \\ (in_x + n_y)\sin(\beta/2) & \cos(\beta/2) + in_z\sin(\beta/2) \end{pmatrix}$$

$$d^1(\beta) = \begin{pmatrix} \frac{1+cos\beta}{2} & -\frac{sin\beta}{\sqrt{2}} & \frac{1-cos\beta}{2} \\ -\frac{sin\beta}{\sqrt{2}} & cos\beta & \frac{sin\beta}{\sqrt{2}} \\ \frac{1-cos\beta}{2} & \frac{sin\beta}{\sqrt{2}} & \frac{1+cos\beta}{2} \end{pmatrix}$$

3.10. Fermi Golden Rule (F04-15)

-To find rate of ionization of an atom due to an external potential.

$$\Gamma_{i\to f} = \sum_f \frac{2\pi}{\hbar}| < f|V|i > |^2\delta(E_f - E_i)$$

-For a plane wave, use $e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta}$
-For an external Electric field, $\hat{V} = -\frac{e}{2}\vec{r}\cdot\vec{E}$. But when you integrate over ϕ, only term that survives is $-\frac{e}{2}E_{0z}r\cos\theta$

$$V(t) = \hat{V}(r)e^{i\omega t} + \hat{V}(r)e^{-i\omega t}$$

$$\int_0^\infty dx x^n e^{-\alpha x} = \frac{n!}{\alpha^{n-1}}$$

-Use $f(k) = \delta(E_f - E_i)$ to determine k's of the ionized particles:

$$\delta(f(k)) = \sum_{zeros-of-f} \frac{\delta(k-k_0)}{|f'(k_0)|}$$

$$d^3n = \frac{V}{(2\pi)^3}d^3k = \frac{4\pi V}{(2\pi\hbar)^3}p^2dp$$

$$\Gamma_{tot} = \sum_f \Gamma_{i\to f} = \int \frac{d^3k}{2\pi}\frac{2\pi}{\hbar}| < f|V|i > |^2\frac{\delta(k-k_0)}{|f'k_0|}$$

3.11. Partial Waves(F01-17)

$$\psi = \frac{U(r)}{r}; -\frac{\hbar^2}{2m}\frac{d^2U}{dr^2} + VU = \frac{\hbar^2k^2}{2m}U$$

$$U(r) = \begin{matrix} sin(kr+\delta_0) & r > R \\ Asinh(Kr) & r < R \end{matrix}$$

$$a = -\frac{\delta_0}{k}$$

$$\sigma = 4\pi a^2$$

$$\sigma = \frac{4\pi}{k^2}\sum_l (2l+1)sin^2\delta_l$$

3.12. Born Approximation (F03-16)

To find cross section of free particle (plane wave) scattering in a potential: Take $k' - k = q$ and find

$$\tilde{V}(q) = \int d^3r V(r)e^{-i\vec{q}\cdot\vec{r}}$$

where $\vec{q}\cdot\vec{r} = qr\cos\theta$. Next find:

$$\frac{d\sigma}{d\Omega} = |\frac{1}{4\pi}\frac{2m}{\hbar^2}\tilde{V}(q)|^2$$

Now take

$$q^2 = k^2 + k'^2 - 2\vec{k}\cdot\vec{k'} = 2k^2(1-\cos\theta)$$

To find

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega}$$

Hint: to do the integral, don't sub in q, Instead, replace $d(cos\theta)$ with dq.

$$\Gamma = (incidentflux) \times (crosssection)$$

3.13. Misc. Facts

-Proton rest mass $938\frac{Mev}{c^2}$
-Electron rest mass $0.511\frac{Mev}{c^2}$

$$<\frac{1}{r}>=\frac{1}{n^2a}$$

-Aharonov-Bohm phase $\theta=\frac{\Phi_m}{\Phi_0}$, where $\Phi_0=\frac{\hbar c}{e}$
-Rotational Hamiltonian, $H_{rot}=\frac{L^2}{2I}$, where $I=\mu r^2$

$$\alpha=\frac{1}{137}=\frac{e^2}{\hbar c}$$

$$\hat{P}\psi_{n,l,m}=(-1)^l\psi_{n,l,m}$$

$$\hbar c\approx 1973eV\AA$$

-Double slit experiment, $d\sin\theta=n\lambda$
-For photon transitions, $\Delta l=\pm1$, $\Delta m=\pm1$
-For Landau levels (F02-18) use $A=\frac{1}{2}B(-y\hat{x}+x\hat{y})$, set $\nabla\cdot A=0$, get $E=\frac{\hbar^2k^2}{2m}+\hbar\omega_L(n+1/2)$
-To find selection rules for quadrupole moment, write in terms of Y_{lm}:

$$Q_{zz}=z^2-\frac{1}{2}r^2=r^2(\cos^2\theta-\frac{1}{2})=\frac{2r^2}{3}P_2\sim Y_{30}$$

4. THERMO/STATS

4.1. Thermo

$$dE=(\frac{\partial E}{\partial S})_{N,V}dS+(\frac{\partial E}{\partial V})_{N,S}dV+(\frac{\partial E}{\partial N})_{V,S}dN$$

$$dE=TdS-PdV+\mu dN$$

$$F=E-TS$$

$$H=E+PV$$

$$G=F+PV=E-TS+PV=\mu N$$

$$(\frac{\partial S}{\partial V})_{N,E}=\frac{P}{T}$$

$$(\frac{\partial S}{\partial N})_{V,E}=-\frac{\mu}{T}$$

$$C_v=T(\frac{\partial S}{\partial T})_{N,V}=(\frac{\partial E}{\partial T})_{N,V}$$

$$C_p=T(\frac{\partial S}{\partial T})_{N,P}=(\frac{\partial H}{\partial T})_{N,P}$$

$$\gamma=1+\frac{2}{DOF}=\frac{C_p}{C_v}$$

-Specific heat, $C=\frac{\Delta Q+\Delta W}{\Delta T}$

	DOF	γ
monotomic	3	5/3
diatomic	5	7/5
polyatomic	6	4/3

$$E=\frac{(\#DOF)}{2}nRT$$

$$\Delta Q=0\rightarrow\Delta(PV^\gamma)=0$$

-Reversible, slow, constant equilibrium process implies $\Delta S = 0$. Spontaneous process, entropy increases. Entropy can never decrease.

$$\Delta S = \int_{T_i}^{T_f} C\frac{dT}{T}$$

$$\texttt{efficiency} = \epsilon = \frac{\Delta W}{\Delta Q} = \frac{Q_1 - Q_2}{Q_1}$$

$$\Delta S = 0 \rightarrow T_f = \sqrt{T_1 T_2}$$

-Latent heat of evaporation:

$$L = h^s - h^l = (\frac{H}{m})^{solid} - (\frac{H}{m})^{liquid}$$

4.2. Microcanonical Ensemble, (fixed E,N,V)

$$S = kln\Omega$$

$$\Omega = \frac{V}{\lambda_T^3}$$

$$\lambda_T = \frac{h}{\sqrt{2\pi mkT}}$$

Two state system:

$$\Omega = \frac{N!}{N_1!N_2!}$$

from Gibb's Paradox

$$S = Nkln[\frac{V}{N\lambda^3}] + \frac{5}{2}Nk$$

4.3. Density of States $a(\epsilon)$

$$\Sigma(P) = \frac{1}{h^3}\int d^3q \int_0^P d^3p = \frac{V}{h^3}\frac{4\pi}{3}P^3$$

$$P = \sqrt{2mE} \rightarrow \Sigma(E)$$

$$a(\epsilon)d\epsilon = \frac{d\Sigma(\epsilon)}{d\epsilon}d\epsilon$$

4.4. Canonical Ensemble, (fixed N,V,T)

-Free energy (F, also written as A):

$$dF = dE - TdS - SdT = -PdV + \mu dN - SdT$$

$$P = -(\frac{\partial F}{\partial V})_{T,N}$$

,

$$S = -(\frac{\partial F}{\partial T})_{N,V}$$

,

$$\mu = -(\frac{\partial F}{\partial N})_{T,V}$$

$$P_r = \frac{exp(-\beta E_r)}{\sum_r exp(-\beta E_r)} = \frac{exp(-\beta E_r)}{Q}$$

$$< lnP_r >= -lnQ - \beta < E_r >= \beta(A - U) = -S/k$$

$$S = -k < lnP_r >= -k\sum_r P_r lnP_r$$

(for microcanonical ensemble, $P_r = \frac{1}{\Omega}$)

$$< E >= -\frac{\partial}{\partial \beta}lnQ$$

$$< (\Delta E >)^2 >=< E^2 > - < E >^2 = -\frac{\partial < E >}{\partial \beta} = kT^2C_v$$

$$F = -kTlnQ_N$$

$$S = -(\frac{\partial F}{\partial T})_{N,V}, P = -(\frac{\partial F}{\partial V})_{N,T}, \mu = (\frac{\partial F}{\partial N})_{V,T}$$

4.5. Finding Partition Function for Canonical Ensembles

$$Q_N = \sum_i g_i e^{-\beta E_i} = \int_0^\infty e^{-\beta E}g(E)dE$$

Classically,

$$Q_N(V,T) = \frac{1}{N!h^{3N}}\int e^{-\beta H(q,p)}d^{3N}qd^{3N}p$$

So for $H = \frac{p^2}{2m}$ you should get $Q_N = \frac{1}{N}(\frac{V}{\lambda_T})^{3N}$

4.5.1. Harmonic Oscillator

Classically, $Q_N = (\frac{1}{\beta\hbar\omega})$. This gives the result of $\bar{E} = NkT$.

(btw, only use $\frac{1}{N!}$ for indistinguishable particles.)

$$g(E) = \frac{1}{(\hbar\omega)^N}\frac{E^{N-1}}{N-1!}$$

Now a for a QM approach, $E_i = (n_i + \frac{1}{2})\hbar\omega$

$$Q_N = (2sinh(\frac{1}{2}\beta\hbar\omega))^{-N}$$

Which approaches the classical result for small $\beta\hbar\omega$

$$< E >= N[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}]$$

Which gives the "0 pt energy"

4.5.2. Paramagnetism

Classically,

$$E_i = -\mu H \sum_i cos\theta_i$$

$$Q_1 = \frac{4\pi sinh(\beta\mu H)}{\beta\mu H}$$

$$M_z = N\mu < cos\theta >= \frac{N}{\beta}\frac{\partial}{\partial H} lnQ_1 \approx \frac{1}{3}N\mu^2 H\beta for(\beta\mu H << 1)$$

$$\chi = (\frac{\partial M}{\partial H})_T$$

(M is analogous to $\bar{E}$ and χ is analogous to C_v)

Quantum Mechanically,

$$\vec{\mu} = g\frac{e}{2mc}\vec{l}$$

4.6. Grand Canonical Ensemble

$$P_{r,s} = \frac{\omega_{r,s}e^{-\alpha N_r - \beta E_s}}{Q_{script}}$$

$$\alpha = \beta\mu$$

$$z = e^{-\alpha}$$

$$\bar{N} = -\frac{\partial}{\partial\alpha} ln[Q_{script}]$$

$$\bar{E} = -\frac{\partial}{\partial\beta} ln[Q_{script}]$$

$$q = ln[Q_{script}] = \frac{S}{k} - \alpha\bar{N} - \beta\bar{E} = \frac{PV}{kT}$$

$$Q_{script} = \sum_{N_r} z^{N_r}\Omega_r, se^{-\beta E_s}$$

Classically,

$$Q_{script} = \frac{1}{1 - zQ_1}$$

$$Q_{script} = \Pi_\epsilon(1 + aze^{-\beta\epsilon})^{-1}$$

$$q = \ln Q_{script} = \mp\sum_\epsilon \ln(1 \mp ze^{-\beta\epsilon})$$

Bose, $z < 1$ and $a = -1$. Fermi, z is arbitrary and $a = +1$.

$$< N >= \sum_\epsilon \frac{1}{z^{-1}e^{\beta\epsilon} + a}$$

$$< E >= \sum_\epsilon \frac{\epsilon}{z^{-1}e^{\beta\epsilon} + a}$$

$$\sum_k \to V\int\frac{d^3k}{(2\pi)^3}$$

4.7. Ideal Bose Gas ($z < 1$)

$$N = \sum_\epsilon \frac{1}{z^{-1}e^{\beta\epsilon} - 1}$$

$$p = \sqrt{2mE} \to g(E) = \frac{2\pi V}{\hbar^3}(2m)^{3/2}E^{1/2}$$

$$N = \frac{2\pi V(2m)^{3/2}}{h^3}\int_0^\infty \frac{E^{1/2}dE}{z^{-1}e^{\beta E}-1} + \frac{z}{1-z} = N_e + N_0$$

$$N_e = \frac{V}{\lambda^3} g_{3/2}(z)$$

$$N_e \leq N_e(1) = \frac{V}{\lambda^3}\zeta(3/2)$$

$$T_c = \frac{1}{\lambda^2}\left[\frac{N}{V\zeta(3/2)}\right]^{2/3}$$

$$N_e = N(\frac{T}{T_c})^{3/2}$$

$$g_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}}{z^{-1}e^x-1}dx = \sum_{l=1}^{\infty}\frac{z^l}{l^\nu}$$

$$g_\nu(1) = \zeta(\nu)$$

$$g_1(z) = \int_0^\infty \frac{dx}{z^{-1}e^x-1} = -\ln(1-z)$$

$$z\frac{\partial}{\partial z}g_\nu(z) = g_{\nu-1}(z)$$

$$q = \ln Q_{script} = -\sum_\epsilon \ln(1-ze^{-\beta\epsilon})$$

$$P = \frac{kT}{V}q = -\frac{kT}{V}\int \frac{d^3k}{(2\pi)^3}\ln(1-e^{-\beta\epsilon})$$

-Given a dispersion relation, $\epsilon(k)$, (F04-17), first find density of states $a(\epsilon)$, then use

$$N = \int_0^\infty \frac{a(\epsilon)d\epsilon}{z^{-1}e^{\beta\epsilon}-1}$$

Then figure out ν to replace with g_ν and plug in $\epsilon = 0$ to get $\frac{z}{1-z}$. $N_e(z=1) \to T_c$

4.8. Fermi Dirac

$$N = \sum_\epsilon \frac{1}{z^{-1}e^{\beta\epsilon}+1}$$

$$\frac{N}{kT} = \frac{g}{\lambda^3}f_{3/2}(z)$$

$$\frac{PV}{kT} = \sum_\epsilon \ln(1+ze^{-\beta\epsilon})$$

$$\frac{P}{kT} = \frac{g}{\lambda^3}f_{5/2}(z)$$

$$f_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}}{z^{-1}e^x+1}dx = \sum_{l=1}^{\infty}(-1)^{l-1}\frac{z^l}{l^\nu} = z - \frac{z^2}{2^\nu}$$

$$z\frac{\partial}{\partial z}f_\nu(z) = f_{\nu-1}(z)$$

Pauli Magnetization:

$$N^+ \sim (\epsilon_E + \mu^* B)^{3/2}, N^+ \sim (\epsilon_E - \mu^* B)^{3/2}$$

$$M = \mu^*(N^+ - N^-)$$

Landau Levels:

$$\epsilon = \frac{\hbar eB}{mc}(j+\frac{1}{2}) + \frac{p_z^2}{2m}$$

To find v_{rms}:

$$<E> = \frac{\int_0^{p_F}\frac{d^p}{(2\pi\hbar)^3}E(p)}{\int_0^{p_F}\frac{d^p}{(2\pi\hbar)^3}} = \frac{\frac{1}{2m}\int_0^{p_F}p^4dp}{\int_0^{p_F}p^2dp}$$

Confining fermions to a parabolic trap, makes a bubble: (F01-20)

$$\mu = \frac{p_F^2}{2m} + \frac{1}{2}m\omega^2r^2 = \frac{1}{2}m\omega^2a^2$$

4.9. White Dwarf Stars

$$p_F = \hbar(3\pi^2 n)^{1/3}$$

$$E = 2V\int_0^{p_F}\frac{d^3p}{(2\pi\hbar)^3}\sqrt{(pc)^2+(mc^2)^2}$$

$$P = -\frac{\partial E}{\partial V}$$

$$u = \frac{d\epsilon}{dp}$$

$$P = \frac{1}{3}n < pu >= \frac{8\pi}{3h^3}\int_0^{p_F} p^3 u dp$$

Outward force $= \frac{dE}{dR} = 4\pi R^2 P(R)$
Outward Pressure $=\frac{1}{3}\frac{N}{V} < pu >$
Inward force $= \frac{\alpha GM^2}{R^2}$
Inward Pressure $= \frac{\alpha}{4\pi}\frac{GM^2}{R^4}$
isothermal compressibility

$$K = -\frac{1}{V}\frac{\partial V}{\partial P}$$

4.10. Ising model

$$\sum_{nn}\sigma_i\sigma_j \approx z\bar{\sigma}\sum_i \sigma_i$$

$$m_0 \approx \pm\sqrt{3}(1 - \frac{T}{T_c})^{1/2}$$

$$N^+ = (\frac{1+m}{2})N = p_+ N$$

$$N^- = (\frac{1-m}{2})N = p_- N$$

Alternatively, minimize $F(m)$

4.11. Van der Waals

$$(P + \frac{a}{V^2})(V - b) = RT$$

Find V_{crit} and T_{crit} when $\frac{\partial P}{\partial V} = \frac{\partial^2 P}{\partial V^2} = 0$

$$\frac{dT}{dV} = -\frac{\partial H/\partial V}{\partial H/\partial T} = 0$$

This gives the inversion temp below which expansion leads to cooling.

4.12. MISC

-The adiabatic invariant $\frac{\mu_0 H}{k_B T}$ can be used for cooling of demagnitization

-Blackbody Radiation:

$$u(\omega)d\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3 d\omega}{e^{\frac{\hbar\omega}{kT}} - 1}$$

$$\bar{E} = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}$$

-Internal Energy is a state variable (path independent) only depends on C and ΔT.

-Convert temperature to Kelvin when comparing $\frac{T_f}{T_i}$, (Add 273).

-Entropy of mixing, multiply partion functions, $z_1 z_2$
-If $Z = \frac{1}{N!}(\frac{V}{\lambda^3})^N$

$$S = -\frac{\partial F}{\partial T} = kNln(\frac{V}{N\lambda^3})$$

-Spread in linewidth $\delta\lambda = \frac{v}{c}\lambda$

-To go from prob density $p_r(z)$ to number density $n(z)$:

$$n(z) = \frac{N}{V}p_r(z)$$

-Now pressure, $P(z) = n(z)kT$ and $\bar{P}\frac{1}{h}\int_0^h P(z)dz$

-For a reaction in equilibrium, $dG = 0$ so chemical potentials must balance.

-For phase transitions, $g = \frac{G}{N}$ must be the same along coexistance curve.

4.13. Math

$$\ln(n!) \approx n\ln n - n$$

$$< (\Delta n >)^2 >\equiv< [n_r - < n_r >]^2 >=< n_r^2 > - < n_r >^2$$

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln(a^b) = b(\ln(a))$$

5. MATH

$$\int \nabla \cdot A dV = \int\int A \cdot \hat{n} dS$$

$$\int \nabla \times A dV = \int\int \hat{n} \times A dS$$

5.1. Cylindrical Coordinates

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\psi} + \frac{\partial\psi}{\partial z}\hat{z}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla\times\vec{v} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \partial_\rho & \partial_\phi & \partial_z \\ v_\rho & rv_\phi & v_z \end{vmatrix}$$

5.2. Spherical Coordinates

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{1}{rsin\theta}\frac{\partial\psi}{\partial\phi}\hat{\psi}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{rsin\theta}\frac{\partial}{\partial\theta}(sin\theta A_\theta) + \frac{1}{rsin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\nabla\times\vec{v} = \frac{1}{r^2 sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ v_r & rv_\theta & r\sin\theta v_\phi \end{vmatrix}$$

5.3. Integrals

$$\int_0^\infty r^n e^{-\alpha r^2} dr = \begin{cases} \sqrt{\frac{\pi}{\alpha}}, & n=0 \\ \frac{1}{4}\frac{\sqrt{\pi}}{\alpha^{3/2}}, & n=2 \\ \frac{3}{8}\frac{\sqrt{\pi}}{\alpha^{5/2}}, & n=4 \\ \frac{15}{16}\frac{\sqrt{\pi}}{\alpha^{7/2}}, & n=6 \end{cases}$$

$$\int_0^\infty x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}$$

$$\int dx \sin^2 kx = \frac{x}{2} - \frac{\cos 2kx}{4k}$$

$$\int dx \cos^2 kx = \frac{x}{2} + \frac{\cos 2kx}{4k}$$

5.4. Expansions

$$e^x = \sum_0^\infty \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + ...$$

$$ln(1+x) = x - \frac{x^2}{x} + \frac{x^3}{3} - ...$$

$$\frac{1}{1-x} = \sum x^n$$

$$\sum_{i=n}^\infty x^i = \frac{1-x^{n+1}}{1-x}$$

$$\sin x = x - \frac{x^3}{6}$$

$$\cos x = 1 - \frac{x^2}{2}$$

$$\tan x = x + \frac{x^3}{3}$$

$$\sinh x = x + \frac{x^3}{6}$$

$$\cosh x = 1 + \frac{x^2}{2}$$

$$\tanh x = x - \frac{x^3}{3}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

$$\frac{1}{2}[\cos v - \cos u] = \sin(\frac{u+v}{2}) sin(\frac{u-v}{2})$$

6. CONTOUR INTEGRATION

$$P \int_{-\infty}^{+\infty} \frac{f(x)}{x - \alpha} dx = i\pi f(\alpha)$$

$$P \int_{-\infty}^{+\infty} \frac{A(x)}{B(x)} dx = \begin{array}{l} 2\pi i \sum Res(\texttt{poles in upper half}) \\ -i\pi \sum Res(\texttt{poles on real axis}) \end{array}$$

Residue of a double pole:

$$Res = [\frac{d}{dz}(z - z_0)^2 f(z)]_{z=z_0}$$

Residue of a nonsimple pole, $f(z) = \frac{\phi(z)}{(z-z_0)^m}$

$$Res = |\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} f(z)|_{z=z_0}$$

$$\ln z = log(|z|e^{i\theta}) = \ln|z| + i\theta$$

$$\int_0^{2\pi} f(\sin\theta, \cos\theta) d\theta$$

Substitute $z = e^{i\theta}$, $\sin\theta = \frac{z-z^{-1}}{2i}$

$$\int_{-\infty}^{+\infty} \frac{cos(x)}{\texttt{trouble}} dx$$

Substitute $cos(x) = \frac{1}{2}(e^{iax} + e^{-iax})$ and do 2 integrals. Jordan's Lemma, if $f(z) \to 0$ as $z \to \infty$, big (pos Im) arc disappears:

$$\int_{-\infty}^{+\infty} e^{iax} f(x) dx$$

$$\int_{-1}^{1} \to \texttt{Expand contour!!}$$

To take residue at ∞ expand integrand for large z... the term over z is the residue.

7. EXPERIMENT/MISC.

Mean Free Path, n = density, σ = cross section

$$l = \frac{1}{n\sigma}$$

Stephan Boltzmann, $\sigma = 5.7x10^{-8}$

$$U = \frac{E}{V} \propto \sigma T^4$$

$$P_{rad} = \sigma T^4 (4\pi R^2)$$

Virial Theorem, $V \propto r^k$

$$2 < T >= k < U >$$

1D Random walk, N steps of length l

$$(\texttt{Path length}_{rms} = \sqrt{N} l$$

Error Analysis

$$\sigma_f = \sqrt{[(\frac{\partial f}{\partial x})_{|(x_0,y_0)} \sigma_x]^2 + [(\frac{\partial f}{\partial y})_{|(x_0,y_0)} \sigma_y]^2} \quad (1)$$